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# The Mittag-Leffler-type function of arbitrary order and its properties

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**Abstract**

In this paper, we introduce a new generalized function of Mittag-Leffler type. We investigate its basic properties, including recurrence relations, differential formulas, integral representations, Euler transform, Laplace transform, Mellin transform, Whittaker transform, and Mellin–Barnes integral representation. We also express it in terms of Fox-Wright function and H-function. Furthermore, we establish fractional integral and differential operators associated with this generalized Mittag-Leffler type function. Several interesting special cases of our main results are derived.

**Keywords:** Mittag-Leffler function; Fox-Wright function; integral transforms; fractional calculus operators.
**Mathematics Subject Classification:** 33E12; 65R10.

## 1 Introduction

The function $E_\alpha(z)$ is named after the renowned Swedish mathematician Gösta Magnus Mittag-Leffler, who defined it using a power series [9]

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha,z\in\mathbb{C},\Re(\alpha)>0. \tag{1.1}$$

First generalizion of the function $E_\alpha(z)$ was introduced by Wiman [22]

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad \alpha,\beta,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0. \tag{1.2}$$

Prabhakar [13] proposed a further generalization of the function $E_\alpha(z)$ in the form

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\quad \alpha,\beta,\gamma,z\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0, \tag{1.3}$$

where $(\gamma)_n$ denotes the Pochhammer symbol defined in terms of the familiar Gamma function $\Gamma$ by (see, e.g., [19])

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1 & (n=0),\\ \gamma(\gamma+1)...(\gamma+n-1) & (n\in\mathbb{N}:=\{1,2,...\}).\end{cases}$$

Moreover, the generalization of $E^{\gamma}_{\alpha,\beta}(z)$ was given by Salim [15], defined as follows:

$$E^{\gamma,\delta}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\,(\delta)_n}z^n, \tag{1.4}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$.

Recently, several generalizations and extensions for Mittag-Leffler functions have been presented and investigated by many authors (see, e.g., [1, 2, 3, 4, 5, 10, 11, 17]). Very recently, Pathan and Bin-Saad [12] introduced a new Mittag-Leffler-type function of arbitrary order, which is a generalization of the Mittag-Leffler function $E_{\alpha,\beta}(z)$. The arbitrary order Mittag-Leffler-type function is defined as

$$E_{\alpha,\beta}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma\left(\beta + \alpha(nj+k)\right)}, \quad \alpha, \beta, z \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, j \geq 1, k \geq 0. \tag{1.5}$$

In the present paper, we introduce and investigate a new generalization of arbitrary order Mittag-Leffler-type function defined as

$$E_{\alpha,\beta,\gamma,\delta}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma\left(\beta + \alpha(nj+k)\right)} z^{nj+k}, \tag{1.6}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ and $j \geq 1$, $k \geq 0$.

Particular cases of $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$

(i) For $\gamma = 1$ and $\delta = 1$, equation (1.6) gives Mittag-Leffler function defined in (1.5)

$$E_{\alpha,\beta,1,1}^{j,k}(z) = \sum_{n=0}^{\infty} \frac{z^{nj+k}}{\Gamma\left(\beta + \alpha(nj+k)\right)} = E_{\alpha,\beta}^{j,k}(z).$$

(ii) For $j = 1$ and $k = 0$, equation (1.6) gives Mittag-Leffler function defined in (1.4)

$$E_{\alpha,\beta,\gamma,\delta}^{1,0}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma\left(\beta + \alpha n\right)} z^n = E_{\alpha,\beta}^{\gamma,\delta}(z).$$

(iii) For $j = 1, k = 0$ and $\delta = 1$, equation (1.6) gives Mittag-Leffler function defined in (1.3)

$$E_{\alpha,\beta,\gamma,1}^{1,0}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!\, \Gamma\left(\beta + \alpha n\right)} z^n = E_{\alpha,\beta}^{\gamma}(z).$$

(iv) For $j = 1, k = 0, \gamma = 1$ and $\delta = 1$, equation (1.6) gives Mittag-Leffler function defined in (1.2)

$$E_{\alpha,\beta,1,1}^{1,0}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(\beta + \alpha n\right)} z^n = E_{\alpha,\beta}(z).$$

(v) For $j = 1, k = 0, \gamma = 1, \delta = 1$ and $\beta = 1$, equation (1.6) gives Mittag-Leffler function defined in (1.1)

$$E_{\alpha,1,1,1}^{1,0}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(\alpha n + 1\right)} z^n = E_{\alpha}(z).$$

The following well-known notations, formulas, and functions are required to prove our main results:
The Beta function is defined as [19]

$$B(\nu, \mu) = \int_0^1 t^{\nu-1} (1-t)^{\mu-1}\, dt, \quad \Re(\nu) > 0, \Re(\mu) > 0, \tag{1.7}$$

or in terms of gamma function as

$$B(\nu, \mu) = \frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}, \quad \nu, \mu \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{1.8}$$

The Fox-Wright function is defined as [20]

$$ {}_p\Psi_q \left[ \begin{array}{c} (d_1, D_1), ..., (d_p, D_p) \\ (e_1, E_1), ..., (e_q, E_q) \end{array} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(d_i + D_i n)}{\prod_{j=1}^{q} \Gamma(e_j + E_j n)} \frac{z^n}{n!}, \tag{1.9} $$

where $d_i, D_i, e_j, E_j, z \in \mathbb{C}$, $\Re(d_i) > 0, \Re(D_i) > 0$, $i = 1, ..., p$, $\Re(e_i) > 0, \Re(E_i) > 0$, $j = 1, ..., q$ and $1 + \Re\left(\sum_{j=1}^{q} E_j - \sum_{i=1}^{p} D_i\right) \geq 0$.

The H-function is given as

$$ H_{P,Q}^{M,N} \left[ z \middle| \begin{array}{c} (A_1, \alpha_1), ..., (A_P, \alpha_P) \\ (B_1, \beta_1), ..., (B_Q, \beta_Q) \end{array} \right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{M} \Gamma(B_j + \beta_j s) \prod_{i=1}^{N} \Gamma(1 - A_i - \alpha_i s)}{\prod_{i=N+1}^{P} \Gamma(A_i + \alpha_i s) \prod_{j=M+1}^{Q} \Gamma(1 - B_j - \beta_j s)} z^{-s} ds, \tag{1.10} $$

where $M, N, P, Q$ are integers such that $0 \leq M \leq Q$, $0 \leq N \leq P$, and the parameters $A_i, B_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}^+ (i = 1, ..., p; j = 1, ..., q)$ with the contour L suitably chosen, and an empty product, if it occurs, is taken to be unity. For more details about the H-function, one can refer to Kilbas and Saigo [6].

The generalized hypergeometric function is given as [14]

$$ {}_pF_q \left[ \begin{array}{c} a_1, ..., a_p \\ b_1, ..., b_q \end{array} ; z \right] = {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!}, \tag{1.11} $$

where the infinite series converges for all $z \in \mathbb{C}$ when $p \leq q$.

The Euler (Beta) transform of the function $f(z)$ is defined as [18]

$$ B\{f(z); \nu, \mu\} = \int_0^1 z^{\nu-1} (1-z)^{\mu-1} f(z) dz, \quad \Re(\nu) > 0, \Re(\mu) > 0. \tag{1.12} $$

The Laplace transform of the function $f(z)$ is defined as [18]

$$ \mathcal{L}\{f(z); s\} = \int_0^{\infty} e^{-sz} f(z) dz, \quad \Re(s) > 0. \tag{1.13} $$

The Mellin transform of the function $f(z)$ is given as [18]

$$ \mathcal{M}\{f(z); s\} = \int_0^{\infty} z^{s-1} f(z) dz = f^*(s), \quad \Re(s) > 0, \tag{1.14} $$

and the inverse Mellin transform is defined as

$$ f(z) = \mathcal{M}^{-1}\{f^*(s); z\} = \frac{1}{2\pi i} \int_L f^*(s) z^{-s} ds, \tag{1.15} $$

where L is a contour of integration that begins at $-i\infty$ and ends at $i\infty$.

The Whittaker transform is defined as [21]

$$ \int_0^{\infty} u^{\nu-1} e^{-\frac{u}{2}} W_{\lambda,\mu}(u) du = \frac{\Gamma\left(\frac{1}{2} + \mu + \nu\right) \Gamma\left(\frac{1}{2} - \mu + \nu\right)}{\Gamma(1 - \lambda + \nu)}, \tag{1.16} $$

where $\Re(\mu \pm \nu) > -\frac{1}{2}$ and $W_{\lambda,\mu}(u)$ is the Whittaker confluent hypergeometric function.

The fractional-order integration and differentiation are defined by the left-sided Riemann-Liouville fractional integral operator $I_{a+}^{\nu}$ and the right-sided Riemann-Liouville fractional integral operator $I_{b-}^{\nu}$, and the corresponding Riemann-Liouville fractional derivative operators $D_{a+}^{\nu}$ and $D_{b-}^{\nu}$, as [20]

$$ \left(I_{a+}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) dt, \quad (\Re(\nu) > 0, x > a), \tag{1.17} $$

$$\left(I_{b-}^{\nu} f\right)(x)=\frac{1}{\Gamma(\nu)} \int_{x}^{b}(t-x)^{\nu-1} f(t) d t, \quad(\Re(\nu)>0, x<b), \tag{1.18}$$

$$\left(D_{a+}^{\nu} f\right)(x)=\left(\frac{d}{d x}\right)^{m}\left(I_{a+}^{m-\nu} f\right)(x), \quad(\Re(\nu)>0, m=[\Re(\nu)]+1) \tag{1.19}$$

and

$$\left(D_{b-}^{\nu} f\right)(x)=(-1)^{m}\left(\frac{d}{d x}\right)^{m}\left(I_{b-}^{m-\nu} f\right)(x), \quad(\Re(\nu)>0, m=[\Re(\nu)]+1), \tag{1.20}$$

where $\Re(\nu)$ denotes the real part of the complex number $\nu \in \mathbb{C}$ and $[\Re(\nu)]$ represents the integral part of $\Re(\nu)$. Here, we recall the left and right-sided Riemann-Liouville fractional integrations of a power function are defined in [7] by

$$\left(I_{0+}^{\nu} t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)}{\Gamma(\lambda+\nu)} x^{\lambda+\nu-1}, \quad(\Re(\nu)>0, \Re(\lambda)>0), \tag{1.21}$$

$$\left(I_{-}^{\nu} t^{\lambda-1}\right)(x)=\frac{\Gamma(1-\nu-\lambda)}{\Gamma(1-\lambda)} x^{\lambda+\nu-1}, \quad(0<\Re(\nu)<1-\Re(\lambda)), \tag{1.22}$$

respectively. The left and right-sided Riemann-Liouville fractional differentiations of a power function are defined, respectively, by (see [7])

$$\left(D_{0+}^{\nu} t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)}{\Gamma(\lambda-\nu)} x^{\lambda-\nu-1}, \quad(\Re(\nu)>0, \Re(\lambda)>0) \tag{1.23}$$

and

$$\left(D_{-}^{\nu} t^{\lambda-1}\right)(x)=\frac{\Gamma(1+\nu-\lambda)}{\Gamma(1-\lambda)} x^{\lambda-\nu-1}, \quad(\Re(\nu)>0, \Re(\lambda)<\Re(\nu)-[\Re(\nu)]). \tag{1.24}$$

## 2 Basic properties

Some basic properties of Mittag-Leffler-type function $E_{\alpha,\beta,\gamma,\delta}^{j,k}(z)$ are presented in this section.

**Theorem 2.1** *If $\alpha,\beta,\gamma,\delta,z \in \mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $j \geq 1$, $k \geq 0$, $k$ even number, then*

$$E_{\alpha,\beta,\gamma,\delta}^{j,k}(-z)=(-1)^{k} E_{2\alpha,\beta,\gamma,\delta}^{j,k/2}(z^{2})+(-1)^{j+k} z^{j} E_{2\alpha,\beta+\alpha j,\gamma,\delta}^{j,k/2}(z^{2}), \tag{2.1}$$

$$E_{\alpha,\beta,\gamma,\delta}^{j,2k}(z)+E_{\alpha,\beta,\gamma,\delta}^{j,2k}(-z)=2 E_{2\alpha,\beta,\gamma,\delta}^{j,k}(z^{2})+\left(1+(-1)^{j+k}\right) z^{j} E_{2\alpha,\beta+\alpha j,\gamma,\delta}^{j,k}(z^{2}), \tag{2.2}$$

*in particular,*

$$E_{2\alpha,\beta}^{\gamma,\delta}(z^{2})=\frac{E_{\alpha,\beta}^{\gamma,\delta}(z)+E_{\alpha,\beta}^{\gamma,\delta}(-z)}{2}. \tag{2.3}$$

**Proof.** If we split the $n$-sum of (1.6) into odd and even indices, we get

$$\begin{aligned}
E_{\alpha,\beta,\gamma,\delta}^{j,k}(-z) &=\sum_{n=0}^{\infty} \frac{(\gamma)_{n}(-z)^{nj+k}}{(\delta)_{n} \Gamma(\beta+\alpha(nj+k))} \\
&=\sum_{n=0}^{\infty} \frac{(\gamma)_{n}(-z)^{2nj+k}}{(\delta)_{n} \Gamma(\beta+\alpha(2nj+k))}+\sum_{n=0}^{\infty} \frac{(\gamma)_{n}(-z)^{2nj+j+k}}{(\delta)_{n} \Gamma(\beta+\alpha(2nj+j+k))}
\end{aligned}$$

$$= (-1)^k \sum_{n=0}^{\infty} \frac{(\gamma)_n (z^2)^{nj+k/2}}{(\delta)_n \Gamma(\beta + 2\alpha(nj + k/2))} + (-1)^{j+k} z^j \sum_{n=0}^{\infty} \frac{(\gamma)_n (z^2)^{nj+k/2}}{(\delta)_n \Gamma(\beta + \alpha j + 2\alpha(nj + k/2))}$$
$$= (-1)^k E^{j,k/2}_{2\alpha,\beta,\gamma,\delta}(z^2) + (-1)^{j+k} z^j E^{j,k/2}_{2\alpha,\beta+\alpha j,\gamma,\delta}(z^2),$$

which is the result (2.1).

Next, in (2.1), if we replace $z$ by $-z$, we get

$$E^{j,k}_{\alpha,\beta,\gamma,\delta}(z) = E^{j,k/2}_{2\alpha,\beta,\gamma,\delta}(z^2) + z^j E^{j,k/2}_{2\alpha,\beta+\alpha j,\gamma,\delta}(z^2). \tag{2.4}$$

Now, adding (2.4) to (2.1) results in

$$E^{j,k}_{\alpha,\beta,\gamma,\delta}(-z) + E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$$
$$= (-1)^k E^{j,k/2}_{2\alpha,\beta,\gamma,\delta}(z^2) + (-1)^{j+k} z^j E^{j,k/2}_{2\alpha,\beta+\alpha j,\gamma,\delta}(z^2) + E^{j,k/2}_{2\alpha,\beta,\gamma,\delta}(z^2) + z^j E^{j,k/2}_{2\alpha,\beta+\alpha j,\gamma,\delta}(z^2),$$

after a little simplification, we get the required desired result (2.2).

On substituting $j = 1$ and $k = 0$ in (2.2), we obtain the desired result (2.3).

**Theorem 2.2** *If $\alpha, \beta, \gamma, \delta, z \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ and $j \geq 1$, $k \geq 0$, $m \in \mathbb{N}$, then*

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega z^\alpha)\right] = z^{\beta-m-1} E^{j,k}_{\alpha,\beta-m,\gamma,\delta}(\omega z^\alpha). \tag{2.5}$$

**Proof.** Using (1.6) and differentiating $m$ times, we have

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega z^\alpha)\right] = \sum_{n=0}^{\infty} \frac{(\gamma)_n\ \omega^{nj+k}}{(\delta)_n\ \Gamma(\beta - m + \alpha(nj+k))} z^{\alpha(nj+k)+\beta-m-1}$$
$$= z^{\beta-m-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n\ \Gamma(\beta - m + \alpha(nj+k))} (\omega z^\alpha)^{nj+k},$$

which leads to the required result (2.5).

**Corollary 2.1** *If we set $j = 1$ and $k = 0$ in equation (2.5), we obtain the following known result, which is due to Salim [15]:*

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta}(\omega z^\alpha)\right] = z^{\beta-m-1} E^{\gamma,\delta}_{\alpha,\beta-m}(\omega z^\alpha). \tag{2.6}$$

**Corollary 2.2** *If we set $\gamma = 1$ and $\delta = 1$ in equation (2.5), we obtain the following known result, which is due to Pathan and Bin-Saad [12]:*

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E^{j,k}_{\alpha,\beta}(\omega z^\alpha)\right] = z^{\beta-m-1} E^{j,k}_{\alpha,\beta-m}(\omega z^\alpha). \tag{2.7}$$

We further utilize (2.5) to derive a differentiation recursion related to fractional values of the parameter $\alpha$.

**Theorem 2.3** *If $\alpha, \beta, \gamma, \delta, z \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ and $j \geq 1$, $k \geq 0$, $m, r \in \mathbb{N}$, $r > m$, then*

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E^{j,k}_{\frac{m}{rj},\beta,\gamma,\delta}(z^{\frac{m}{rj}})\right] = z^{\beta-1} E^{j,k}_{\frac{m}{rj},\beta,\gamma,\delta}(z^{\frac{m}{rj}}) + z^{\beta-1} \sum_{n=1}^{r} \frac{(\gamma)_n\ z^{-\frac{m}{rj}(nj-k)}}{(\delta)_n\ \Gamma\left(\beta - \frac{m}{rj}(nj-k)\right)}. \tag{2.8}$$

**Proof.** Taking the left-hand side of (2.8),

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E^{j,k}_{\frac{m}{rj},\beta,\gamma,\delta}(z^{\frac{m}{rj}})\right]$$

$$
\begin{aligned}
&= \left(\frac{d}{dz}\right)^{m} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n\ \Gamma\left(\beta+\frac{m}{rj}(nj+k)\right)} z^{\frac{m}{rj}(nj+k)+\beta-1} \\
&= z^{\beta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n\ \Gamma\left(\beta-m+\frac{m}{rj}(nj+k)\right)} z^{\frac{m}{rj}(nj+k)-m} \\
&= z^{\beta-1} \sum_{n=-r}^{\infty} \frac{(\gamma)_n}{(\delta)_n\ \Gamma\left(\beta+\frac{m}{rj}(nj+k-rj)\right)} z^{\frac{m}{rj}(nj+k-rj)} \\
&= z^{\beta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n\ z^{\frac{m}{rj}(nj+k-rj)}}{(\delta)_n\ \Gamma\left(\beta+\frac{m}{rj}(nj+k-rj)\right)} + z^{\beta-1} \sum_{n=-r}^{-1} \frac{(\gamma)_n\ z^{\frac{m}{rj}(nj+k-rj)}}{(\delta)_n\ \Gamma\left(\beta+\frac{m}{rj}(nj+k-rj)\right)} \\
&= z^{\beta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n\ z^{\frac{m}{rj}(nj-rj+k)}}{(\delta)_n\ \Gamma\left(\beta+\frac{m}{rj}(nj-rj+k)\right)} + z^{\beta-1} \sum_{n=1}^{r} \frac{(\gamma)_n\ z^{-\frac{m}{rj}(nj+rj-k)}}{(\delta)_n\ \Gamma\left(\beta-\frac{m}{rj}(nj+rj-k)\right)}.
\end{aligned}
$$

After simplification, we get the desired result:

$$
\left(\frac{d}{dz}\right)^{m} \left[z^{\beta-1} E^{j,k}_{\frac{m}{rj},\beta,\gamma,\delta}(z^{\frac{m}{rj}})\right] = z^{\beta-1} E^{j,k}_{\frac{m}{rj},\beta,\gamma,\delta}(z^{\frac{m}{rj}}) + z^{\beta-1} \sum_{n=1}^{r} \frac{(\gamma)_n\ z^{-\frac{m}{rj}(nj-k)}}{(\delta)_n\ \Gamma\left(\beta-\frac{m}{rj}(nj-k)\right)}.
$$

**Corollary 2.3** *If we set $j=1$ and $k=0$ in equation (2.8), we obtain the following new result:*

$$
\left(\frac{d}{dz}\right)^{m} \left[z^{\beta-1} E^{\gamma,\delta}_{\frac{m}{r},\beta}(z^{\frac{m}{r}})\right] = z^{\beta-1} E^{\gamma,\delta}_{\frac{m}{r},\beta}(z^{\frac{m}{r}}) + z^{\beta-1} \sum_{n=1}^{r} \frac{(\gamma)_n\ z^{-\frac{m}{r}n}}{(\delta)_n\ \Gamma\left(\beta-\frac{m}{r}n\right)}. \tag{2.9}
$$

**Corollary 2.4** *If we set $r=1$ in equation (2.8), we obtain the following new result:*

$$
\left(\frac{d}{dz}\right)^{m} \left[z^{\beta-1} E^{j,k}_{\frac{m}{j},\beta,\gamma,\delta}(z^{\frac{m}{j}})\right] = z^{\beta-1} E^{j,k}_{\frac{m}{j},\beta,\gamma,\delta}(z^{\frac{m}{j}}) + \frac{\gamma}{\delta} \frac{z^{\beta-m+\frac{m}{j}k-1}}{\Gamma\left(\beta-m+\frac{m}{j}k\right)}. \tag{2.10}
$$

Further, if $\gamma=1$ and $\delta=1$, equation (2.10) coincides with the result in [12].

**Theorem 2.4** *If $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $j\geq 1$, $k\geq 1$, then*

$$
E^{j,k}_{\alpha,\beta,\gamma,\delta}(z) = \frac{z^k}{(b-a)^{\beta+\alpha k-1}\,\Gamma(\alpha k)} \int_a^b (t-a)^{\beta-1}\,(b-t)^{\alpha k-1}\, E^{\gamma,\delta}_{\alpha j,\beta}\left(\left(\frac{t-a}{b-a}\right)^{\alpha j} z^j\right) dt. \tag{2.11}
$$

**Proof.**

$$
\begin{aligned}
&\int_a^b (t-a)^{\beta-1}\,(b-t)^{\alpha k-1}\, E^{\gamma,\delta}_{\alpha j,\beta}\left(\left(\frac{t-a}{b-a}\right)^{\alpha j} z^j\right) dt \\
&= \int_a^b (t-a)^{\beta-1}\,(b-t)^{\alpha k-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n\ z^{nj}}{(\delta)_n\ \Gamma(\beta+\alpha nj)} \left(\frac{t-a}{b-a}\right)^{\alpha nj} dt \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n\ z^{nj}}{(\delta)_n\ \Gamma(\beta+\alpha nj)} (b-a)^{-\alpha nj} \int_a^b (t-a)^{\beta+\alpha nj-1}\,(b-t)^{\alpha k-1}\, dt \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_n\ z^{nj}}{(\delta)_n\ \Gamma(\beta+\alpha nj)} (b-a)^{\beta+\alpha k-1}\, B(\beta+\alpha nj, \alpha k) \\
&= (b-a)^{\beta+\alpha k-1}\,\Gamma(\alpha k) \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n\ \Gamma(\beta+\alpha(nj+k))} z^{nj} \\
&= z^{-k}\,(b-a)^{\beta+\alpha k-1}\,\Gamma(\alpha k) E^{j,k}_{\alpha,\beta,\gamma,\delta}(z).
\end{aligned}
$$

Next, we begin by recalling that the Gaussian hypergeometric function is defined as follows:

$$ {}_2F_1\left(a,b;c;z\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\quad \left(|z|<1\right). \tag{2.12}$$

**Theorem 2.5** *If $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $j\geq 1$, $k\geq 0$, then*

$$E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)=\frac{z^k}{2\pi i}\int_H e^t\ t^{-(\beta+\alpha k)}{}_2F_1\left(\gamma,1;\delta;\frac{z^j}{t^{\alpha j}}\right)dt, \tag{2.13}$$

where $H$ denotes the Hankel contour encircling the negative real axis and circulation in the positive direction.
**Proof.** Applying definition (2.12), we obtain

$$\int_H e^t\ t^{-(\beta+\alpha k)}{}_2F_1\left(\gamma,1;\delta;\frac{z^j}{t^{\alpha j}}\right)dt=\sum_{n=0}^{\infty}\frac{(\gamma)_n\ z^{nj}}{(\delta)_n}\int_H e^t\ t^{-(\beta+\alpha j+\alpha k)}dt.$$

Using Hankel's representation of the reciprocal Gamma function (see, [21])

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_H e^t\ t^{-z}dt.$$

Therefore, the integral will become,

$$\begin{aligned}\int_H e^t\ t^{-(\beta+\alpha k)}{}_2F_1\left(\gamma,1;\delta;\frac{z^j}{t^{\alpha j}}\right)dt&=\sum_{n=0}^{\infty}\frac{(\gamma)_n\ z^{nj}}{(\delta)_n}\frac{2\pi i}{\Gamma\left(\beta+\alpha(nj+k)\right)}\\&=2\pi i\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\ \Gamma\left(\beta+\alpha(nj+k)\right)}z^{nj}\\&=\frac{2\pi i}{z^k}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(z\right).\end{aligned}$$

# 3 Representation of $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ in terms of other functions

In this section, we express $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ in terms of the Fox-Wright function, Mellin-Barnes integral and H-function.

The generalized Mittag-Leffler function of arbitrary order $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ can be written as

$$E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\Gamma\left(\beta+\alpha(nj+k)\right)}z^{nj+k}=\frac{z^k\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+n\right)\Gamma\left(1+n\right)}{\Gamma\left(\delta+n\right)\Gamma\left(\beta+\alpha(nj+k)\right)}\frac{z^{nj}}{n!}.$$

Utilizing equation (1.9), $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ can be expressed using the Fox-Wright function as follows:

$$\begin{aligned}E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)&=\frac{z^k\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+n\right)\Gamma\left(1+n\right)}{\Gamma\left(\delta+n\right)\Gamma\left(\beta+\alpha k+\alpha nj\right)}\frac{(z^j)^n}{n!}\\&=\frac{z^k\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}{}_2\Psi_2\left[\begin{array}{c}(\gamma,1),(1,1)\\(\delta,1),(\beta+\alpha k,\alpha j)\end{array}\middle|\,z^j\right].\end{aligned}\tag{3.1}$$

Now, to express $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ in terms of the H-function, we begin by representing $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ as a Mellin-Barnes integral, as stated in the following theorem.

**Theorem 3.1** *For any $z\in\mathbb{C}$ with $|arg(z)|<\pi$, the function $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ can be expressed using the following Mellin-Barnes integral:*

$$E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)=\frac{z^k\Gamma(\delta)}{2\pi i\ \Gamma(\gamma)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-s)}{\Gamma(\delta-s)\Gamma(\beta+\alpha k-\alpha js)}(-z^j)^{-s}ds, \tag{3.2}$$

*where the contour of integration $L$ joins $-i\infty$ to $+i\infty$, and splitting all the poles at $s=-n$, $(n=0,1,2,\ldots)$ to the left and the poles at $s=n+1$ and at $s=\gamma+n$, $(n=0,1,2,\ldots)$ to the right.*

**Proof.** By evaluating the integral on the right-hand side of (3.2), as the sum of residues at the poles $s=-n$, $(n \in \mathbb{N}_0)$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-s)}{\Gamma(\delta-s)\Gamma(\beta+\alpha k-\alpha js)}(-z^j)^{-s}ds \\
&=\sum_{n=0}^{\infty}\operatorname*{Res}_{s=-n}\left[\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-s)}{\Gamma(\delta-s)\Gamma(\beta+\alpha k-\alpha js)}(-z^j)^{-s}\right] \\
&=\sum_{n=0}^{\infty}\lim_{s\to -n}\left[(s+n)\,\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-s)}{\Gamma(\delta-s)\Gamma(\beta+\alpha k-\alpha js)}(-z^j)^{-s}\right] \\
&=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)(-z^j)^n}{\Gamma(\delta+n)\Gamma(\beta+\alpha k+\alpha jn)}\lim_{s\to -n}\left[(s+n)\,\Gamma(s)\Gamma(1-s)\right] \\
&=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)(-z^j)^n}{\Gamma(\delta+n)\Gamma(\beta+\alpha k+\alpha jn)}\lim_{s\to -n}\left[\frac{\pi(s+n)}{sin\pi s}\right] \\
&=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)(-1)^n(-z^j)^n}{\Gamma(\delta+n)\Gamma(\beta+\alpha k+\alpha jn)} \\
&=\frac{z^{-k}\Gamma(\gamma)}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\;\Gamma(\beta+\alpha(nj+k))}z^{nj+k} \\
&=\frac{z^{-k}\Gamma(\gamma)}{\Gamma(\delta)}E^{j,k}_{\alpha,\beta,\gamma,\delta}(z).
\end{aligned}$$

Using equation (1.10), Theorem 3.1 gives

$$E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)=\frac{z^k\Gamma(\delta)}{\Gamma(\gamma)}H^{1,2}_{2,3}\left[-z^j\,\middle|\,\begin{matrix}(0,1),(1-\gamma,1)\\(0,1),(1-\delta,1),(1-\beta-\alpha k,\alpha j)\end{matrix}\right]. \tag{3.3}$$

# 4 Integral transforms

In this section, we establish certain useful integral transforms like Euler transform, Laplace transform, Mellin transform, Whittaker transform.

**Theorem 4.1** *(Euler transform): If $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\nu)>0$, $\Re(\mu)>0$, $\Re(\sigma)>0$ and $j\geq 1$, $k\geq 0$, then*

$$\begin{aligned}
&\int_0^1 z^{\nu-1}(1-z)^{\mu-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(xz^{\sigma})\,dz \\
&=\frac{x^k\Gamma(\mu)\Gamma(\delta)}{\Gamma(\gamma)}{}_3\Psi_3\left[\begin{matrix}(1,1),(\gamma,1),(\nu+\sigma k,\sigma j)\\(\delta,1),(\beta+\alpha k,\alpha j),(\nu+\mu+\sigma k,\sigma j)\end{matrix}\,\middle|\,x^j\right].
\end{aligned} \tag{4.1}$$

**Proof.**

$$\begin{aligned}
&\int_0^1 z^{\nu-1}(1-z)^{\mu-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(xz^{\sigma})\,dz \\
&=\int_0^1 z^{\nu-1}(1-z)^{\mu-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\;(xz^{\sigma})^{nj+k}}{(\delta)_n\;\Gamma(\beta+\alpha(nj+k))}dz.
\end{aligned}$$

After interchanging the integration and summation orders of the equation above, we get

$$\begin{aligned}
&\int_0^1 z^{\nu-1}(1-z)^{\mu-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}(xz^{\sigma})\,dz \\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n\;x^{nj+k}}{(\delta)_n\;\Gamma(\beta+\alpha(nj+k))}\int_0^1 z^{\nu+\sigma nj+\sigma k-1}(1-z)^{\mu-1}\,dz
\end{aligned}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\ \Gamma\left(\beta+\alpha(nj+k)\right)}\frac{\Gamma\left(\mu\right)\Gamma\left(\nu+\sigma k+\sigma nj\right)}{\Gamma\left(\nu+\mu+\sigma k+\sigma nj\right)}x^{nj+k}.$$

Simplification yields the desired result

$$\begin{aligned}&\int_0^1 z^{\nu-1}\left(1-z\right)^{\mu-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(xz^{\sigma}\right)dz\\&=\frac{x^k\Gamma\left(\mu\right)\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}{}_3\Psi_3\left[\begin{array}{c}(1,1),(\gamma,1),(\nu+\sigma k,\sigma j)\\(\delta,1),(\beta+\alpha k,\alpha j),(\nu+\mu+\sigma k,\sigma j)\end{array}\middle|\,x^j\right].\end{aligned}$$

If $j=1$ and $k=0$, equation (4.1) coincides with the result in [15].

**Theorem 4.2** *(Laplace transform): If $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\sigma)>0$, $\Re(p)>0$, $\Re(s)>0$ and $j\geq 1$, $k\geq 0$, then*

$$\int_0^{\infty}z^{p-1}e^{-sz}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(xz^{\sigma}\right)dz=\frac{s^{-(p+\sigma k)}x^k\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}{}_3\Psi_2\left[\begin{array}{c}(1,1),(\gamma,1),(p+\sigma k,\sigma j)\\(\delta,1),(\beta+\alpha k,\alpha j)\end{array}\middle|\left(\frac{x}{s^{\sigma}}\right)^j\right].\tag{4.2}$$

**Proof.** From definition (1.6), we have

$$\int_0^{\infty}z^{p-1}e^{-sz}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(xz^{\sigma}\right)dz=\sum_{n=0}^{\infty}\frac{(\gamma)_n\ x^{nj+k}}{(\delta)_n\ \Gamma\left(\beta+\alpha(nj+k)\right)}\int_0^{\infty}z^{p+\sigma nj+\sigma k-1}e^{-sz}dz.$$

Now, using equation (1.13), we get

$$\begin{aligned}&\int_0^{\infty}z^{p-1}e^{-sz}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(xz^{\sigma}\right)dz\\&=\frac{s^{-(p+\sigma k)}x^k\Gamma\left(\delta+n\right)}{\Gamma\left(\gamma\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\gamma+n\right)\Gamma\left(p+\sigma k+\sigma nj\right)s^{-\sigma nj}x^{nj}}{\Gamma\left(\delta+n\right)\Gamma\left(\beta+\alpha k+\alpha nj\right)}\\&=\frac{s^{-(p+\sigma k)}x^k\Gamma\left(\delta+n\right)}{\Gamma\left(\gamma\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(1+n\right)\Gamma\left(\gamma+n\right)\Gamma\left(p+\sigma k+\sigma nj\right)s^{-\sigma nj}x^{nj}}{n!\ \Gamma\left(\delta+n\right)\Gamma\left(\beta+\alpha k+\alpha nj\right)},\end{aligned}$$

considering (1.9), we obtain (4.2).

**Corollary 4.1** *Substituting $\gamma=\delta=1$ in (4.2), we find*

$$\int_0^{\infty}z^{p-1}e^{-sz}E^{j,k}_{\alpha,\beta}\left(xz^{\sigma}\right)dz=\frac{x^k}{s^{(p+\sigma k)}}{}_2\Psi_1\left[\begin{array}{c}(1,1),(p+\sigma k,\sigma j)\\(\beta+\alpha k,\alpha j)\end{array}\middle|\left(\frac{x}{s^{\sigma}}\right)^j\right].\tag{4.3}$$

If $j=1$ and $k=0$, equation (4.2) coincides with the result in [15].

**Theorem 4.3** *(Mellin transform): If $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(s)>0$ and $j\geq 1$, $k\geq 0$, then*

$$\mathcal{M}\left[(-\omega z)^{-\frac{k}{j}}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left((-\omega z)^{\frac{1}{j}}\right);s\right]=\frac{\Gamma(\delta)\Gamma(s)\Gamma(1-s)\Gamma(\gamma-s)}{\Gamma(\gamma)\Gamma(\delta-s)\Gamma(\beta+\alpha k-\alpha js)}\omega^{-s}.\tag{4.4}$$

**Proof.** Based on Theorem 3.1, we have

$$\begin{aligned}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left((-\omega z)^{\frac{1}{j}}\right)&=\frac{(-\omega z)^{\frac{k}{j}}\Gamma(\delta)}{2\pi i\ \Gamma(\gamma)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-s)}{\Gamma(\delta-s)\Gamma(\beta+\alpha k-\alpha js)}(\omega z)^{-s}ds\\&=\frac{(-\omega z)^{\frac{k}{j}}\Gamma(\delta)}{2\pi i\ \Gamma(\gamma)}\int_L f^*(s)z^{-s}ds,\end{aligned}\tag{4.5}$$

where

$$f^*(s)=\frac{\Gamma(s)\Gamma(1-s)\Gamma(\gamma-s)}{\Gamma(\delta-s)\Gamma(\beta+\alpha k-\alpha js)}\omega^{-s}.$$

Now, making use of the formulas in (1.14), 1.15) and (4.5) directly leads to (4.4).

If $j=1$ and $k=0$, equation (4.4) coincides with the result in [15].

**Theorem 4.4** *(Whittaker transform): If $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\sigma)>0$, $\Re(\nu)>0$ and $j\geq 1$, $k\geq 0$, then*

$$\int_0^\infty u^{\nu-1}e^{-\frac{1}{2}tu}W_{\lambda,\mu}(tu)E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(\omega u^\sigma\right)du$$
$$=\frac{\omega^k t^{-(\nu+\sigma k)}\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}{}_4\Psi_3\left[\begin{array}{c}(1,1),(\gamma,1),(\frac{1}{2}+\mu+\nu+\sigma k,\sigma j),(\frac{1}{2}-\mu+\nu+\sigma k,\sigma j)\\(\delta,1),(\beta+\alpha k,\alpha j),(1-\lambda+\nu+\sigma k,\sigma j)\end{array}\middle|\left(\frac{\omega}{t^\sigma}\right)^j\right]. \tag{4.6}$$

**Proof.** Letting $tu=r$, we obtain

$$\frac{1}{t}\int_0^\infty\left(\frac{r}{t}\right)^{\nu-1}e^{-\frac{r}{2}}W_{\lambda,\mu}(r)E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(\omega\left(\frac{r}{t}\right)^\sigma\right)dr$$
$$=\omega^k t^{-(\alpha+\sigma k)}\sum_{n=0}^\infty\frac{(\gamma)_n}{(\delta)_n\ \Gamma\left(\beta+\alpha(nj+k)\right)}\left(\frac{\omega}{t^\sigma}\right)^{nj}\int_0^\infty r^{\nu+\sigma nj+\sigma k-1}e^{-\frac{r}{2}}W_{\lambda,\mu}(r)dr$$
$$=\frac{\omega^k t^{-(\alpha+\sigma k)}\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}\sum_{n=0}^\infty\frac{\Gamma\left(\gamma+n\right)\Gamma\left(\frac{1}{2}+\mu+\nu+\sigma k+\sigma nj\right)\Gamma\left(\frac{1}{2}-\mu+\nu+\sigma k+\sigma nj\right)}{\Gamma\left(\delta+n\right)\Gamma\left(\beta+\alpha k+\alpha nj\right)\Gamma\left(1-\lambda+\nu+\sigma k+\sigma nj\right)}\left(\frac{\omega^j}{t^{\sigma j}}\right)^n$$
$$=\frac{\omega^k t^{-(\alpha+\sigma k)}\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}\sum_{n=0}^\infty\frac{\Gamma(1+n)\Gamma\left(\gamma+n\right)\Gamma\left(\frac{1}{2}+\mu+\nu+\sigma k+\sigma nj\right)\Gamma\left(\frac{1}{2}-\mu+\nu+\sigma k+\sigma nj\right)}{\Gamma\left(\delta+n\right)\Gamma\left(\beta+\alpha k+\alpha nj\right)\Gamma\left(1-\lambda+\nu+\sigma k+\sigma nj\right)}$$
$$\times\frac{\left(\frac{\omega^j}{t^{\sigma j}}\right)^n}{n!},$$

from (1.9), we get the assertion (4.6).

**Corollary 4.2** *Substituting $j=1$ and $k=0$ in (4.6), we find*

$$\int_0^\infty u^{\nu-1}e^{-\frac{1}{2}tu}W_{\lambda,\mu}(tu)E^{\gamma,\delta}_{\alpha,\beta}\left(\omega u^\sigma\right)du$$
$$=\frac{t^{-\nu}\Gamma\left(\delta\right)}{\Gamma\left(\gamma\right)}{}_4\Psi_3\left[\begin{array}{c}(1,1),(\gamma,1),(\frac{1}{2}+\mu+\nu,\sigma),(\frac{1}{2}-\mu+\nu,\sigma)\\(\delta,1),(\beta,\alpha),(1-\lambda+\nu,\sigma)\end{array}\middle|\frac{\omega}{t^\sigma}\right]. \tag{4.7}$$

## 5 Fractional calculus operators

In this section, we derive some interesting formulas of $E^{j,k}_{\alpha,\beta,\gamma,\delta}(z)$ associated with the operators of Riemann- Liouville fractional integral and derivative.

**Theorem 5.1** *Let $\nu,\alpha,\beta,\gamma,\delta,\omega\in\mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $j\geq 1$, $k\geq 0$, $x>a$. Let $I^\nu_{a+}$ be the left-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I^\nu_{a+}\left[(t-a)^{\beta-1}E^{j,k}_{\alpha,\beta,\gamma,\delta}\left(\omega(t-a)^\alpha\right)\right]\right)(x)=(x-a)^{\beta+\nu-1}E^{j,k}_{\alpha,\beta+\nu,\gamma,\delta}\left(\omega(x-a)^\alpha\right). \tag{5.1}$$

**Proof.** By using definitions (1.6) and (1.17) and applying the following result:

$$\left(I^\nu_{a+}\left[(t-a)^{\beta-1}\right]\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\nu)}(x-a)^{\beta+\nu-1},$$

we arrive at (5.1).

**Theorem 5.2** *Let $\nu,\alpha,\beta,\gamma,\delta,\omega\in\mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $j\geq 1$, $k\geq 0$. Let $I^\nu_-$ be the right-sided operator of Riemann-Liouville fractional integral. Then*

$$\left(I^\nu_-\left[t^{-\nu-\beta}E^{j,k}_{\alpha,\beta,\gamma,\delta}(\omega t^{-\alpha})\right]\right)(x)=x^{-\beta}E^{j,k}_{\alpha,\beta+\nu,\gamma,\delta}(\omega x^{-\alpha}). \tag{5.2}$$

**Proof.** Applying definitions (1.6) and (1.18) and using the following result:

$$\left(I_{-}^{\nu} t^{-\nu-\beta}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\nu)} x^{-\beta},$$

we get (5.2).

**Theorem 5.3** *Let $\nu, \alpha, \beta, \gamma, \delta, \omega \in \mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ and $j \geq 1$, $k \geq 0$, $x>a$. Let $D_{a+}^{\nu}$ be the left-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_{a+}^{\nu}\left[(t-a)^{\beta-1} E_{\alpha, \beta, \gamma, \delta}^{j, k}\left(\omega(t-a)^{\alpha}\right)\right]\right)(x)=(x-a)^{\beta-\nu-1} E_{\alpha, \beta-\nu, \gamma, \delta}^{j, k}\left(\omega(x-a)^{\alpha}\right). \tag{5.3}$$

**Proof.** We have

$$\begin{aligned}
&\left(D_{a+}^{\nu}\left[(t-a)^{\beta-1} E_{\alpha, \beta}^{j, k}\left(\omega(t-a)^{\alpha}\right)\right]\right)(x) \\
&=\left(D_{a+}^{\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{n} \omega^{n j+k}}{(\delta)_{n} \Gamma(\beta+\alpha(n j+k))}(t-a)^{\beta+\alpha(n j+k)-1}\right)(x) \\
&=\sum_{n=0}^{\infty} \frac{(\gamma)_{n} \omega^{n j+k}}{(\delta)_{n} \Gamma(\beta+\alpha(n j+k))}\left(D_{a+}^{\nu}\left[(t-a)^{\beta+\alpha(n j+k)-1}\right]\right)(x) \\
&=(x-a)^{\beta-\nu-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n} \omega^{n j+k}}{(\delta)_{n} \Gamma(\beta-\nu+\alpha(n j+k))}(x-a)^{\alpha(n j+k)} \\
&=(x-a)^{\beta-\nu-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{(\delta)_{n} \Gamma(\beta-\nu+\alpha(n j+k))}\left[\omega(x-a)^{\alpha}\right]^{n j+k} \\
&=(x-a)^{\beta-\nu-1} E_{\alpha, \beta-\nu, \gamma, \delta}^{j, k}\left(\omega(x-a)^{\alpha}\right).
\end{aligned}$$

This completes the proof of the Theorem 5.3.

**Theorem 5.4** *Let $\nu, \alpha, \beta, \gamma, \delta, \omega \in \mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\beta)>[\Re(\nu)]+1$ and $j \geq 1$, $k \geq 0$. Let $D_{-}^{\nu}$ be the right-sided operator of Riemann-Liouville fractional derivative. Then*

$$\left(D_{-}^{\nu}\left[t^{\nu-\beta} E_{\alpha, \beta, \gamma, \delta}^{j, k}\left(\omega t^{-\alpha}\right)\right]\right)(x)=x^{-\beta} E_{\alpha, \beta-\nu, \gamma, \delta}^{j, k}\left(\omega x^{-\alpha}\right). \tag{5.4}$$

**Proof.** We have

$$\begin{aligned}
&\left(D_{-}^{\nu}\left[t^{\nu-\beta} E_{\alpha, \beta, \gamma, \delta}^{j, k}\left(\omega t^{-\alpha}\right)\right]\right)(x) \\
&=\left(D_{-}^{\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{n} \omega^{n j+k}}{(\delta)_{n} \Gamma(\beta+\alpha(n j+k))} t^{\nu-\beta-\alpha(n j+k)}\right)(x) \\
&=\sum_{n=0}^{\infty} \frac{(\gamma)_{n} \omega^{n j+k}}{(\delta)_{n} \Gamma(\beta+\alpha(n j+k))}\left(D_{-}^{\nu} t^{\nu-\beta-\alpha(n j+k)}\right)(x) \\
&=x^{-\beta} \sum_{n=0}^{\infty} \frac{(\gamma)_{n} \omega^{n j+k}}{(\delta)_{n} \Gamma(\beta-\nu+\alpha(n j+k))} x^{-\alpha(n j+k)} \\
&=x^{-\beta} \sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{(\delta)_{n} \Gamma(\beta-\nu+\alpha(n j+k))}\left(\omega x^{-\alpha}\right)^{n j+k} \\
&=x^{-\beta} E_{\alpha, \beta-\nu, \gamma, \delta}^{j, k}\left(\omega x^{-\alpha}\right).
\end{aligned}$$

This completes the proof of the Theorem 5.4.

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